Exact solution for front propagation in a one-dimensional epidemic model

Leonard M. Sander
Physics Dept. & Michigan Center for Theoretical Physics
The University of Michigan

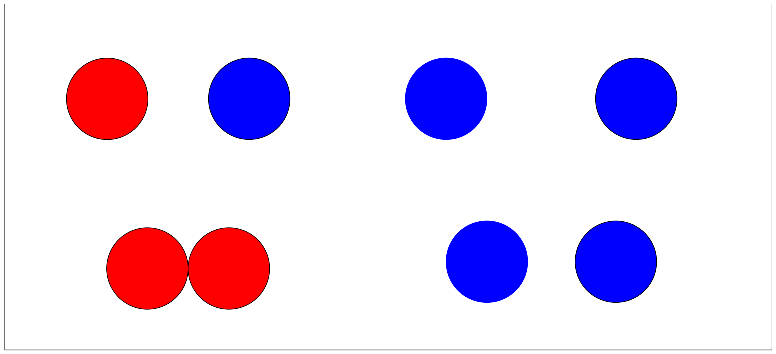
Collaborators: **C. Warren, E. Somfai*, G. Mikus**
*Current address: Instituut Lorentz, Leiden, NL-2333

Published papers:

C. Warren, E. Somfai. and L. M. Sander
Velocity of front propagation in 1-dimensional autocatalytic reactions
Brazilian Journal of Physics 30, 157-162 (2000).

G. Mikus, C. P. Warren, and L.M. Sander
Fluctuation Effects in An Epidemic Model
Phys. Rev. E, 63, 056103 (2001)

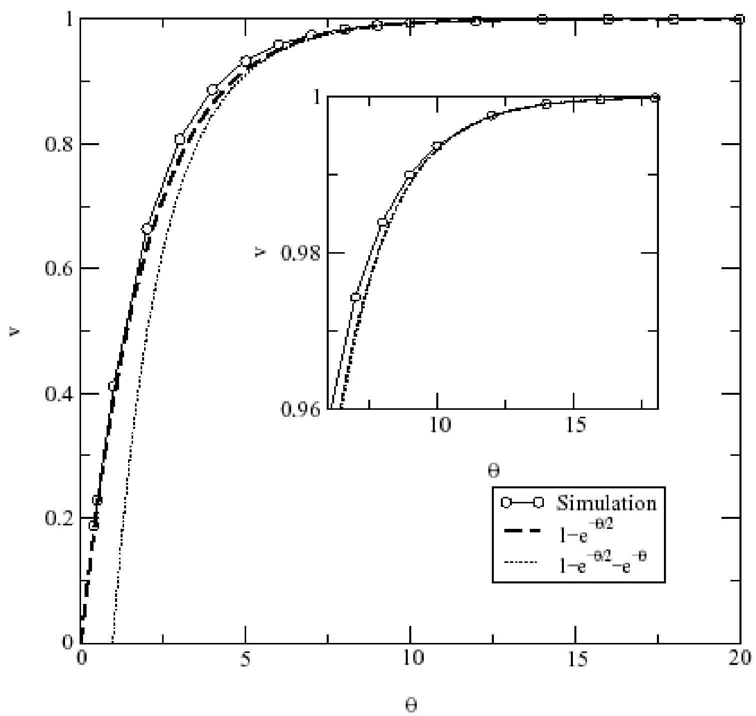
- We study the discrete model of the irreversible reaction $A + B \to 2A$ in one dimension, which is a version of the MSB model:
  J. Mai, I.M. Sokolov and A. Blumen, Phys.Rev.Lett. vol. 77, 4462 (1996).
  This can also be regarded as the simplest way to look at the spread of an epidemic.



- We are able to solve the model exactly in one dimension for low concentrations.
- We find that in the low-concentration limit the average velocity of propagation approaches $D\theta/2$ where θ is the concentration and D the diffusion coefficient.
- The front propagation is entirely dominated by fluctuations in the density: the front spends most of its time *pinned behind gaps in the density.*
  Estimate: $L \sim 1/\theta \quad \tau \sim L^2/D \quad L/\tau = D/L \sim D\theta$
- As a result, continuum modeling *breaks down completely* for this reaction.

Discrete Model

- Consider a 1d lattice of length L populated with random walkers randomly distributed with concentration θ. The leftmost particle is of type A, and all of the other particles are of type B.
- The particles walk randomly and all the particles move simultaneously. Any number of particles are allowed to occupy a site. If a B particle encounters or passes an A particle, it becomes an A particle.
- The rightmost A particle defines the propagation front, and we are interested in the velocity of this front.



Continuum mean field limit

The concentration of A particles should be described by the Fisher-KPP equation.
R. A. Fisher, Annals of Eugenics **7**, 355 (1937);
A. Kolmogorov, I. Petrovsky, and N. Piscounov, Moscow Univ. Bull. Math. **A1**, 1 (1937).
Write a conventional reaction-diffusion equation:

$$\begin{aligned}\partial_t a &= D\Delta a + kab \\ &= D\Delta a + ka(\theta - a),\end{aligned}$$

Here k is a rate constant, D the diffusion constant. and we have used the fact that on average $a + b = \theta$.

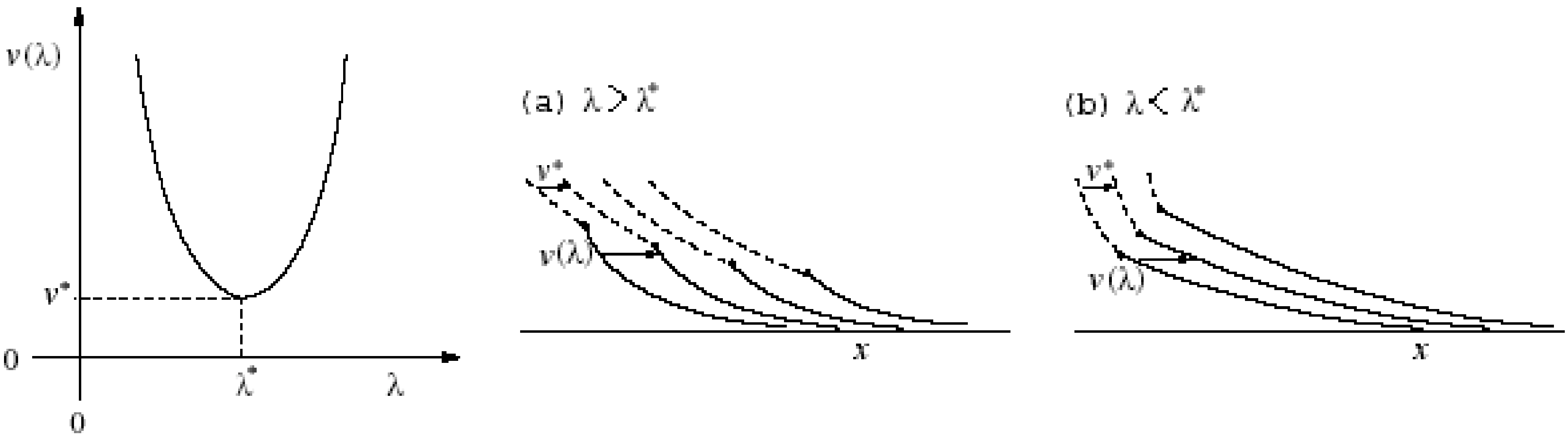
In one dimension, the last equation is the FK equation
$\partial_t \phi = \partial_{xx}\phi + \phi(1-\phi)$
after changing variables.

More than you ever wanted to know about the FK equation

Recent review: D. Panja, cond-mat/0307363

The FK equation describes a ‘pulled’ front. Standard approach:

- Make the substitution $\xi = x - vt$
- Look at $\xi >> 1$, and try $\phi \propto e^{-\lambda\xi}$. This gives $v = \lambda + 1/\lambda$

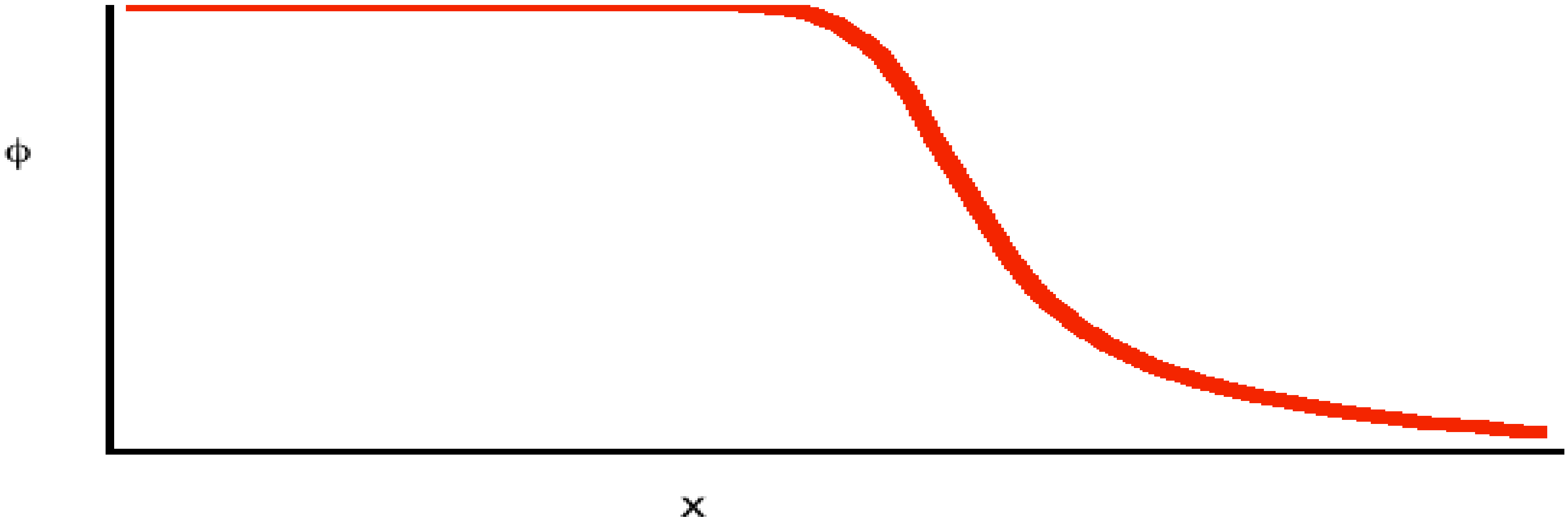


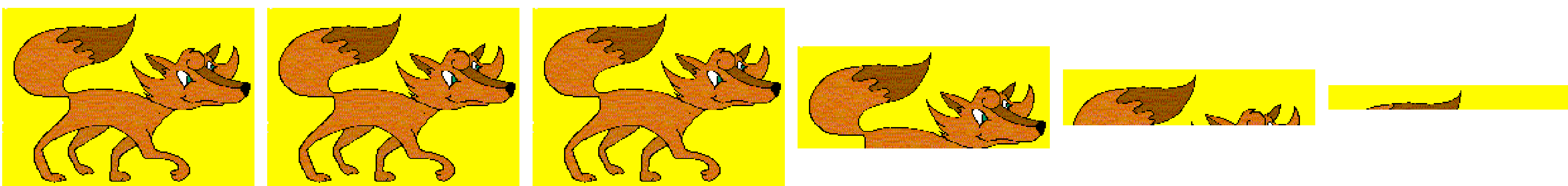
If the initial condition is ***bounded*** the front approaches the minimum speed $v^* = 2$, or, in the original units, $v^* = 2\sqrt{kD\theta}$.

FK equation with discreteness

Note that the velocity is determined where ϕ is very small.

For example, the equation has been used to model the spread of rabies in foxes.





The velocity depends on the behavior of *nano-foxes*!

FK equation, continued

Moreover, simulations of the discrete MSB model showed that in 1d
$v \propto \theta$ for small θ, NOT $v \propto \sqrt{\theta}$

Thus $v << v^*$.

Partial answer, E. Brunet and B. Derrida, Phys. Rev. E **56**, 2597 (1997)
Replace the discreteness by a *cutoff on the growth term*, so that

$$\phi(1-\phi) \rightarrow \phi(1-\phi)\Theta(\phi-\epsilon)$$

(We should think of $\epsilon \sim 1/\theta$.)

The equation gives the analytic result $v \sim v^* - K/\ln^2(\theta)$ where K is a constant.

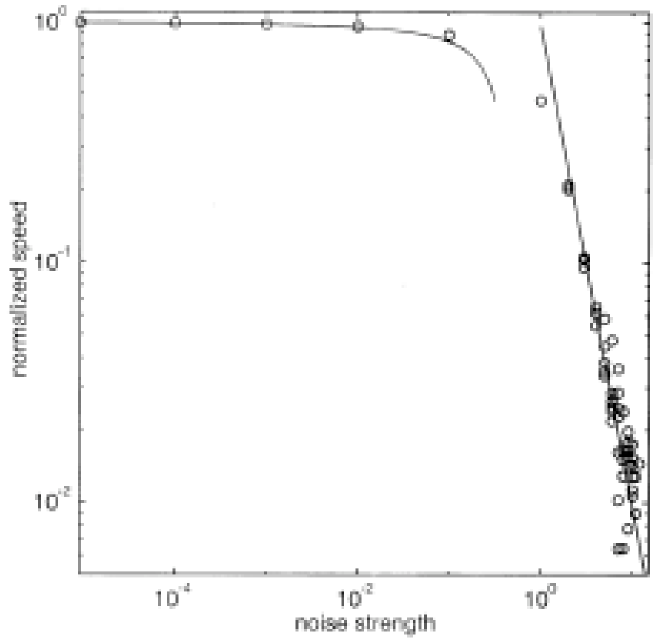
Particle models exist which interpolate between discrete behavior and and the FK equation and find the $\ln^{-2}$ correction.
Brunet and Derrida; D. A. Kessler, Z. Ner and L. M. Sander, Phys. Rev. E. **58**, 107 (1998).
These are for the case $\theta >> 1$, small reaction rate, k.

Fluctuations

Brunet and Derrida represented a ***density fluctuation effect***; cf. the ***stochastic*** FK equation:

$$\partial_t \phi = \partial_{xx}\phi + \phi(1-\phi) + \epsilon^{1/2}\sqrt{\phi(1-\phi)}\,\eta(x,t)$$

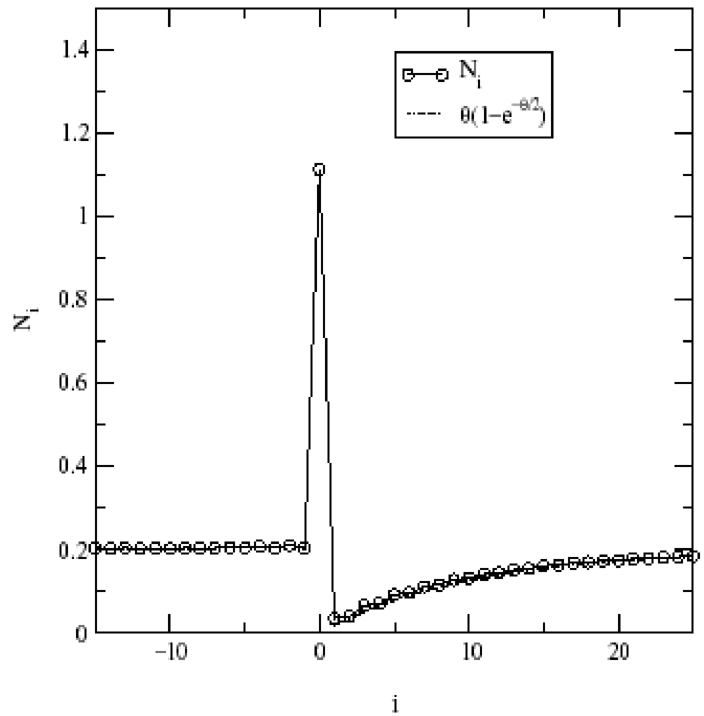
L. Pechenik and H. Levine, Phys. Rev. E 59, 3893 (1999);
C. R. Doering, C. Mueller and P. Smereka, Physica A 325, 243 (2003).



Log-log plot of $v/v*$ in the sFKPP equation on a discrete lattice as a function of noise strength $\epsilon^{1/2}$. Open circles: simulation data for the sFKPP equation, solid lines: the theoretical expressions (Derrida & Brunet) at weak noise and D/ϵ at strong noise.

Simulation: Depletion zone

- In the simulation, the particles follow a simple random walk and thus are Poisson distributed.
- However, taking into account particle types and following the front, the distribution of particles near the front is *not* so distributed.



Average density of particles from simulation for various sites around the front for $\theta = 0.2$. Ignoring the fit for the moment, the density (conditioned on there being a front at $i = 0$), is depleted to the right of the front.

Matching condition

MSB analyzed the velocity in an approximate fashion, using the Smoluchowski approach. In this method, centered in the rest frame of the front, B particles diffuse toward the front. The number density n follows the one-dimensional diffusion equation in the frame moving with velocity v,

$$\frac{\partial n}{\partial t} - vn' = n'', \tag{1}$$

where the diffusion constant $D = 1$. Assuming stationarity, the time derivative vanishes, and the boundary conditions $n(\pm\infty) = \theta$, $n'(\pm\infty) = 0$ and $n(0^+) = 0$ lead to:

$$\begin{aligned} n(x) &= \theta \quad \text{if } x < 0 \\ &= (1 - e^{-vx})\theta \quad \text{if } x \geq 0. \end{aligned}$$

We will see later that $v \propto \theta$ so that,
as $x \to 0^+$, n is order θ^2.
This distribution agrees well with simulation.

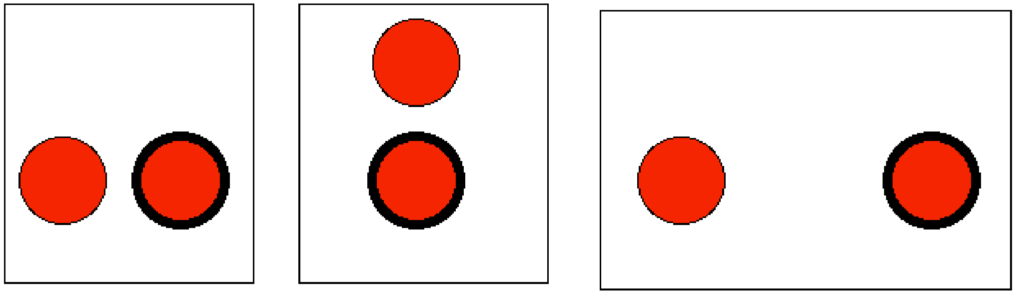
However, this analysis does not determine v.

Master equation for small θ

For $\theta \ll 1$ consider a region containing the front particle and the "second" particle, i.e., that nearest the front. The size of the region will be $\sim 1/\theta$.

Define a coordinate system in the rest frame of the front particle, whose position is defined to be at $i = 0$. The number density of the second particle at site i at time t is $n_i(t)$.

Contributions to the average velocity occur only when the second particle is one behind the front ($i = -1$) or on the front ($i = 0$). For all other positions, the front particle undergoes an unbiased random walk, and the front does not move (on average).



When $i = -1$, with proper renaming of particles, the front will move forward one step with probability $1/2$, stay the same with probability $1/4$ and move back one step with probability $1/4$. Thus, given $i = -1$, the average velocity is $v = 1/2 - 1/4 = 1/4$.

When $i = 0$, the front will move forward one with probability $3/4$ and move back one with probability $1/4$, and the average velocity is $v = 1/2$. Thus,

$$v(t) = \frac{1}{4}n_{-1}(t) + \frac{1}{2}n_0(t). \tag{2}$$

Master equation: Dynamics & Stationary Solution

Each particle performs a random walk so that:

1. If the particle is at $i = 0$ at time t, $n_{-2}(t+1) = 1/2$ and $n_0(t+1) = 1/2$.
2. If the particle is at $i = -1$ at time t, $n_{-1}(t+1) = 3/4$ and $n_{-3}(t+1) = 1/4$.
3. If the particle is at $i < -1$ or $i > 0$, $n_{i+2}(t+1) = 1/4$, $n_i(t+1) = 1/2$ and $n_{i-2}(t+1) = 1/4$.

- For positions away from the front, $i < -2$ or $i > 2$, from 3: $n_i(t+1) = n_i(t)/2 + n_{i+2}(t)/4 + n_{i-2}(t)/4$. Thus, if n is stationary: $n_i = (n_{i-2} + n_{i+2})/2$. That is, n_i is linear in i far from the front.
- In like manner, for positions around the front

$$\begin{aligned} n_{-2} &= n_{-4}/2 + n_0 \\ n_{-1} &= n_{-3} + n_1 \\ n_0 &= (n_{-2} + n_2)/2 \\ n_1 &= n_3/2 \\ n_2 &= n_4/2 \end{aligned} \tag{3}$$

Solution

Since n_i is linear in i far from the front, we only need the slope.

We have made the approximation of only one nonfront particle, which is only valid in the region $1/\theta$ around the front.

Outside this region, we must match to the continuum solution of MSB

$$n(x) = \theta \text{ if } x < 0 \qquad (1 - e^{-vx})\theta \text{ if } x > 0.$$

To order θ, this means:
$n = \theta$ behind the front
$n = 0$ in front.

Thus, to first order in θ, the solution to the equations above is:

$$\begin{aligned} n_i &= \theta \qquad i < 0 \\ &= \theta/2 \quad i = 0 \\ &= 0 \qquad i > 0 \end{aligned}$$

From above, $v(t) = \frac{1}{4}n_{-1}(t) + \frac{1}{2}n_0(t)$. Thus:

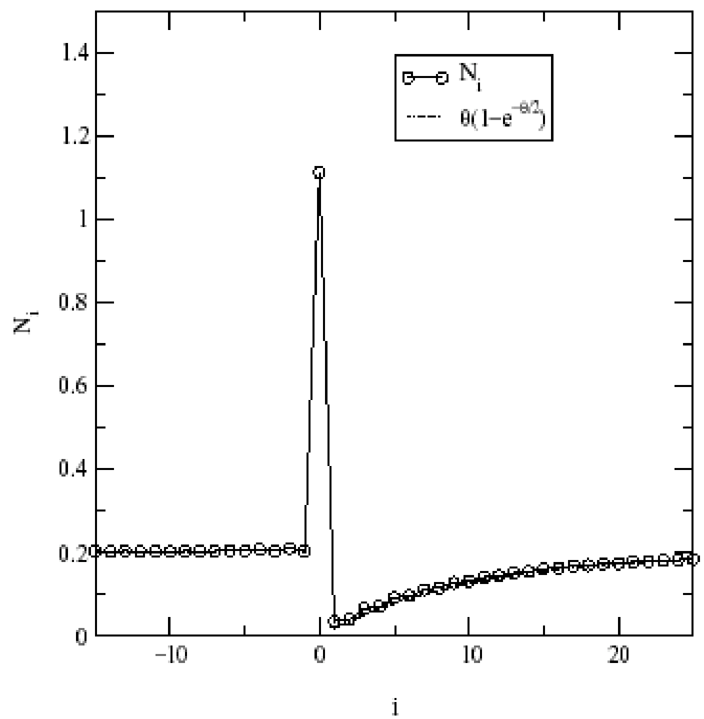
$$v = \theta/2.$$

Density near the front

Note that including the front particle, to first order, the stable ***total*** number density distribution is

- $N_i = \theta$ for $i < 0$
- $N_i = 1 + \theta/2$ for $i = 0$,
- $N_i = 0$ for $i > 0$.

That is we have average concentration to the left of the front, enhancement at the front, and a depleted zone to the right, in agreement with simulations.



Further developments: 2d

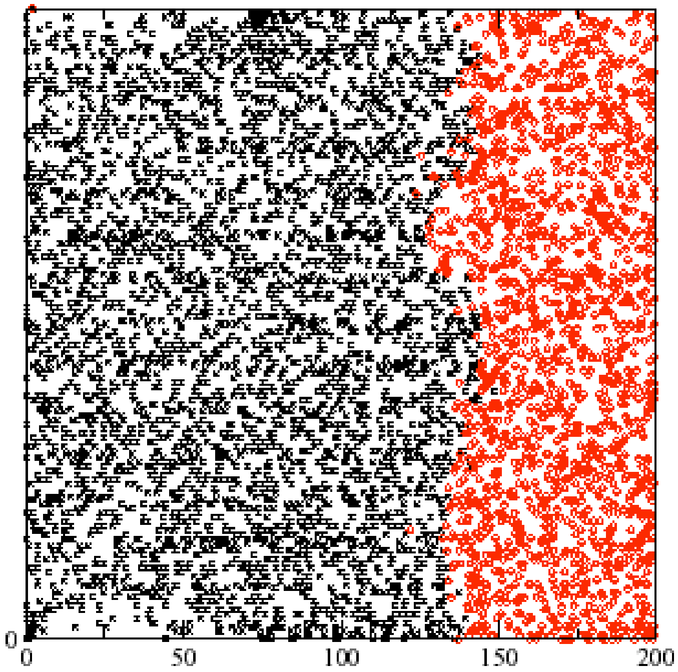
We have generalized the model to two dimensions.

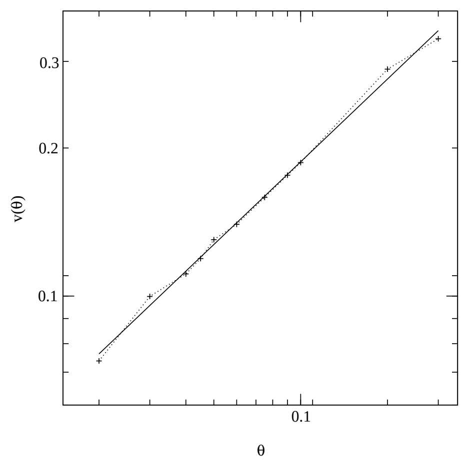
G. Mikus, C. P. Warren, and L.M. Sander, *Fluctuation Effects in An Epidemic Model* Phys. Rev. E, 63, 056103 (2001)

We expect *front wandering* to give rise to KPZ-like behavior on large scales.

J. Riordan, C. Doering, D. Ben-Avraham, PRL 75, 565 (1995); R. Goodman, D. Graff, L. M. Sander, P. Leroux-Hugon, and E. Clement, PRE 52, 5904 (1995); L. M. Sander and S. V. Ghaisas, Physica A 233, 629 (1996)

In 2d, we find a depletion zone, and $v \sim \theta^{0.6}$, significantly different from the mean field result. If we assume the front is self-affine, we measure scaling exponents of $\alpha = 0.84 \pm 0.03$ and $\beta = 0.344 \pm 0.004$, significantly *different* KPZ. This may be a crossover.





Summary

- We have shown that in low concentration the dynamics of $A + B \rightarrow 2A$ are significantly different from what would be expected from mean field theory, even in 2d.
- We have seen that in 1d the behavior of v at low concentration can be traced to the depletion zone to the right of the front. Near the front the distribution of particles is very different from the Poisson distribution, and the motion of the front is dominated by the depletion.
- In the low concentration limit, to order θ, the velocity is simply $v = \theta/2$.
- For large θ, the velocity is approximated by $1 - e^{-\theta/2}$, and the distribution is quite close to the truncated Poisson.