

4. Additional Examples of Reaction Diffusion Problems

(from Mathematical Biology, by Murray) A generalization of Fisher's equation, is the equation

$$\partial_t u(x,t) = \partial_x(u^m \partial_x u(x,t)) + u^p(1-u)^q \text{ for } m,p,q = \text{positive parameters}$$

More general still is the equation

$$\partial_t u(x,t) = \partial_x(D(u) \partial_x u(x,t)) + f(u)$$

but without specific information about the functions $D(u), f(u)$ analytic solutions for this equation are not possible. Moreover, by clever choices of the parameters, we can qualitatively mimic more general D and f . Therefore we will consider the less general equation and we will begin with the case where $m = 1 = p = q$.

Density Dependent Diffusion

Consider the equation

$$\partial_t u(x,t) = \partial_x(u \partial_x u(x,t)) + u(1-u). \quad (4.1)$$

The density dependent diffusivity $D(u) = u$ implies that the rate proportionality of flux to gradient increases as u increases. This means that the flow from crowded regions to less crowded regions proceeds at a higher rate as the crowding increases.

If we suppose $u(x,t) = U(x-ct) = U(z)$, then

$$(U(z)U'(z))' + cU'(z) + U(1-U) = 0.$$

The associated dynamical system is

$$\begin{aligned} U'(z) &= V(z), \\ U(z)V'(z) &= -cV(z) - V(z)^2 - U(1-U) \end{aligned} \quad (4.2)$$

where $U(z) \rightarrow 1$ as $z \rightarrow -\infty$, $U(z) \rightarrow 0$ as $z \rightarrow \infty$, and (we hope) $U'(z) < 0$ all z . We note that the second equation in the system (4.2) is singular at $U = 0$. This singularity can be removed by defining a new variable, s , given by

$$\frac{d}{ds} = U \frac{d}{dz} \quad \text{i.e., } z'(s) = U(z).$$

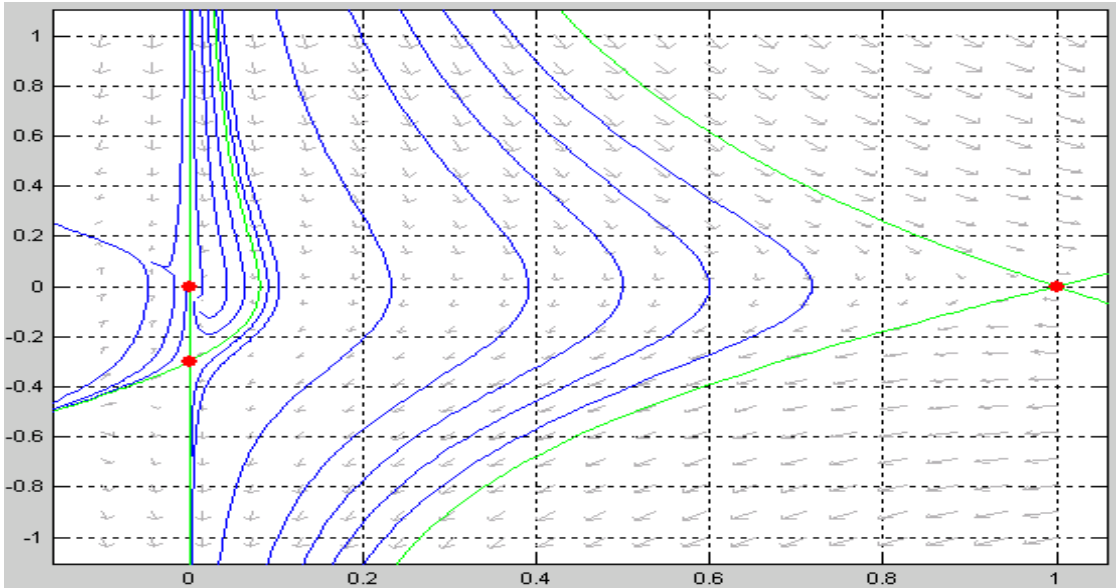
Then (4.2) becomes

$$\begin{aligned} U'(s) &= UV \\ V'(s) &= -cV - V^2 - U(1-U), \end{aligned} \quad (4.3)$$

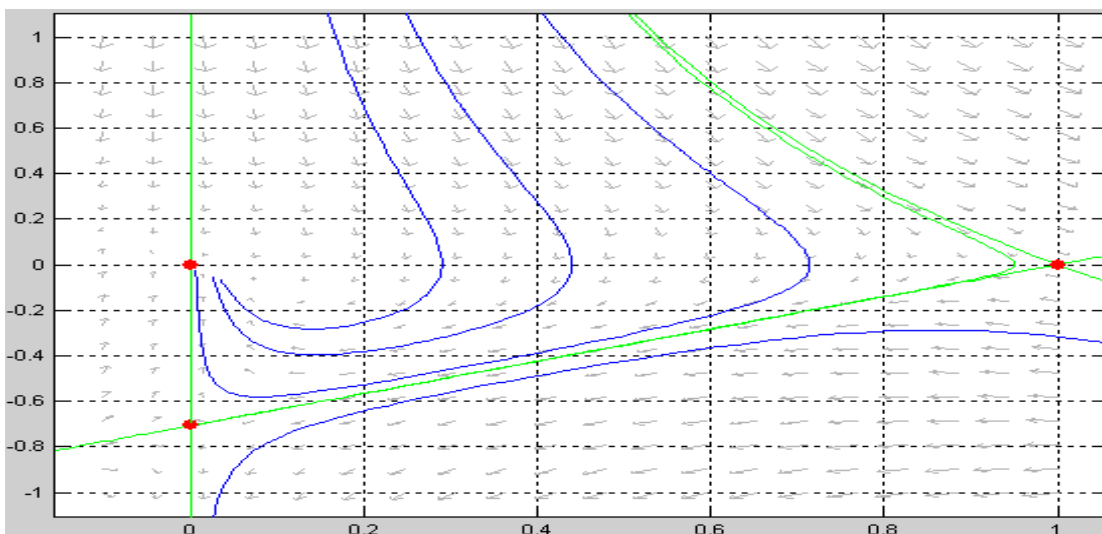
with critical points at $(U, V) = (0, 0)$, $(1, 0)$, and $(0, -c)$. Then

$$J(U, V) = \begin{bmatrix} V & U \\ 2U - 1 & -c - 2V \end{bmatrix}$$

and it follows that: $(0, 0)$ = a stable node, and $(1, 0)$, $(0, -c)$ are both saddles. For c small, say $c = .3$ we have



we see that there is no orbit joining $(1, 0)$ to either of the other critical points. For a slightly larger value of c , say $c = .707$, we have



i.e., in this case there is an orbit joining the saddle at $(1, 0)$ to the saddle at $(0, -c)$. Note that this trajectory tends to $U = 0$, $V = U' = -c$ as $s \rightarrow \infty$. This means that the graph of

U versus z terminates at a point z_0 on the z -axis and $U(z) = 0$ for $z \geq z_0$. The derivative is discontinuous at this point as it jumps from $V(z_0^-) = -c$, to $V(z_0^+) = 0$. The trajectory in this case is a straight line joining $(1, 0)$ to $(0, -c)$, i.e., $V = -c_*(1 - U)$ where $c = c_*$ denotes the critical value of c where this orbit occurs. It follows from the system (4.3) that on this orbit

$$\frac{dV}{dU} = \frac{-cV - V^2 - U(1 - U)}{UV}$$

hence if $V = -c_*(1 - U)$ then we can solve for c_* to obtain $c_* = \frac{1}{\sqrt{2}}$. Then

$$U'(z) = -\frac{1}{\sqrt{2}}(1 - U(z)) \text{ and } U(-\infty) = 1, U(z_0) = 0$$

implies

$$U(z) = 1 - \exp\left(\frac{z - z_0}{\sqrt{2}}\right) \text{ if } z < z_0$$

$$= 0 \text{ if } z > z_0$$

For $c > c_*$ we have an orbit joining $(1, 0)$ to $(0, 0)$ and this results in the usual sort of travelling wave.

Other exercises we could try would include examining the effect of increasing m , p and q above the value 1, testing the stability of such solutions and trying to obtain perturbation approximations for these solutions.

Having considered some examples of TW solutions to scalar reaction diffusion equations, we will consider now the existence of TW solutions to systems of such equations.

Rabies Epidemic in Foxes

Consider the following simple model for the spread of rabies in a population of foxes. Let

$S(x, t)$ = the number of susceptible foxes at location x at time t

$I(x, t)$ = the number of infected foxes at location x at time t .

Then the following equations

$$\partial_t S(x, t) = -rS(x, t)I(x, t)$$

$$\partial_t I(x, t) = rS(x, t)I(x, t) - aI(x, t) + D\partial_{xx}I(x, t),$$

assert that the number of uninfected foxes decreases at a rate that is jointly proportional to the numbers of infected and uninfected foxes. The proportionality factor is denoted by $r > 0$. The second equation asserts that the number of infected foxes increases in the corresponding way but there is also a loss due to deaths of infected foxes. The death rate is

denoted by $a > 0$. Finally, the number of infected foxes is also affected by movement of these foxes which is reflected in the diffusive term in the second equation.

If $S(x, 0) = S_0 > 0$, then we can reduce these equations to dimensionless terms by letting

$$t = \frac{\tau}{rS_0}, \quad x = \sqrt{\frac{D}{rS_0}}, \quad b = \frac{a}{rS_0},$$

and

$$\sigma(z, \tau) = \frac{S(x, t)}{S_0}, \quad \eta(z, \tau) = \frac{I(x, t)}{S_0}.$$

Then

$$\partial_\tau \sigma(z, \tau) = -\sigma \eta$$

$$\partial_\tau \eta(z, \tau) = \sigma \eta - b \eta + \partial_{zz} \eta.$$

We are looking for solutions $\sigma(z, \tau) = f(z - c\tau)$, $\eta(z, \tau) = g(z - c\tau)$ such that

$$\sigma \rightarrow 1 \text{ and } \eta \rightarrow 0 \text{ as } z \rightarrow \infty \quad (\text{no infections ahead of the wave})$$

$$\sigma' \rightarrow 0 \text{ and } \eta \rightarrow 0 \text{ as } z \rightarrow -\infty \quad (\text{no infections behind the wave and } \sigma = \text{constant}).$$

These conditions correspond to an pulse of infection propagating into an initially uninfected population and leaving behind a (possibly) positive number of survivors. It is assumed that all infected foxes eventually die.

These assumptions lead to

$$-c f'(z) = -f(z) g(z)$$

$$-c g'(z) = f(z) g(z) - b g(z) + g''(z).$$

Then
$$g(z) = c \frac{f'(z)}{f(z)}$$

and
$$-c g'(z) - g''(z) = c f'(z) - bc \frac{f'(z)}{f(z)}.$$

Then
$$-c g(z) - g'(z) = c f(z) - bc \log f(z) + C_0,$$

and the condition as $z \rightarrow \infty$ implies $c + C_0 = 0$. Then

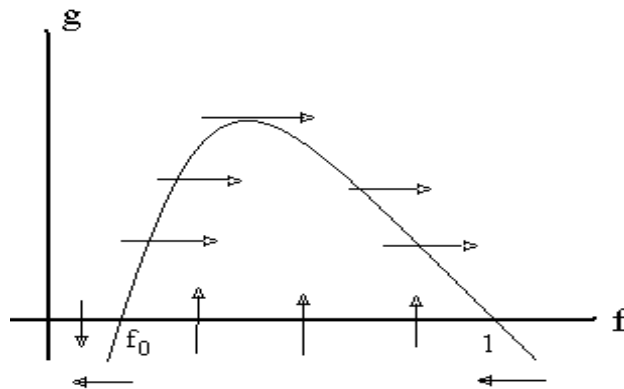
$$-c g(z) - g'(z) = c f(z) - bc \log f(z) - c,$$

and

$$f'(z) = \frac{1}{c} f(z)g(z),$$

$$g'(z) = -cf(z) + bc \log f(z) + c - cg(z).$$

Note that $f' = 0$ on the axes, $f = 0$, $g = 0$, while $g' = 0$ on $g = b \log f - f + 1$. These are the so called "null clines" where the derivatives vanish. The critical points of the dynamical system are located at the intersections of these null clines; i.e., at $f = 1$, $g = 0$ and at $f = f_0$, $g = 0$, for some f_0 , $0 < f_0 < 1$. The direction field near the null clines is shown on the following sketch.



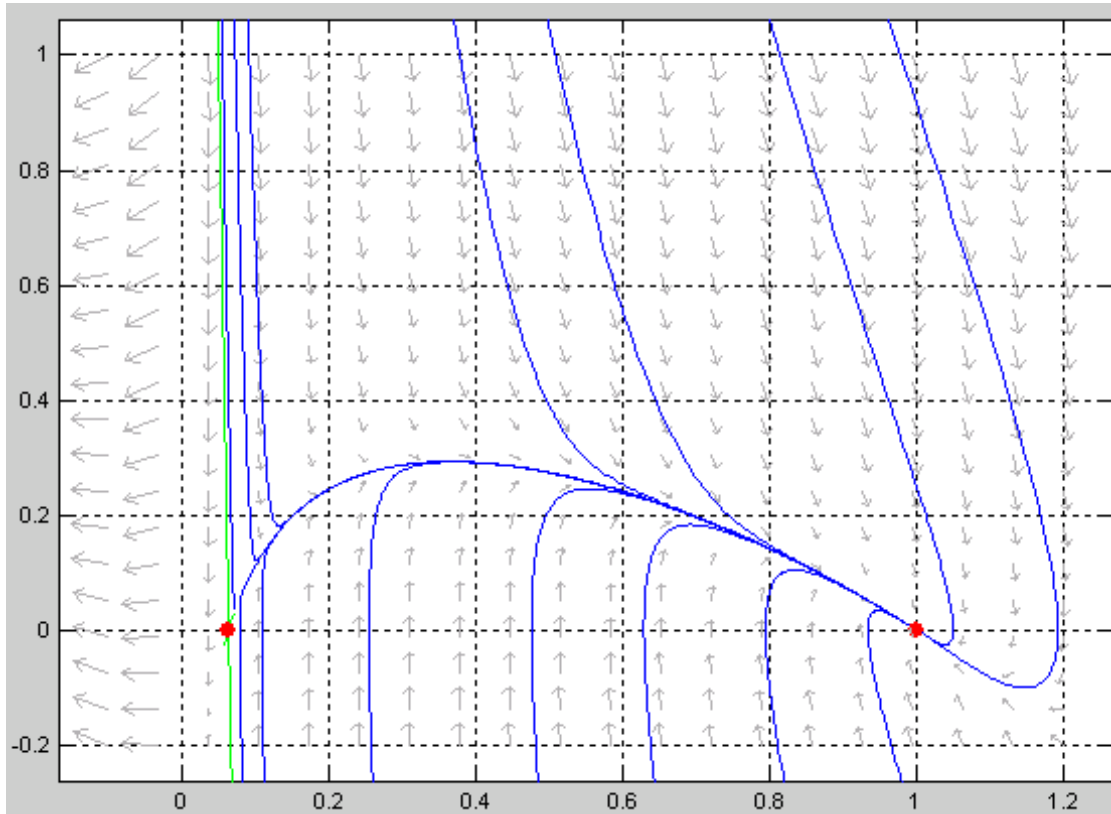
Now the Jacobian of the system is

$$J(f, g) = \begin{bmatrix} \frac{g}{c} & \frac{f}{c} \\ \frac{bc}{f} - c & -c \end{bmatrix}$$

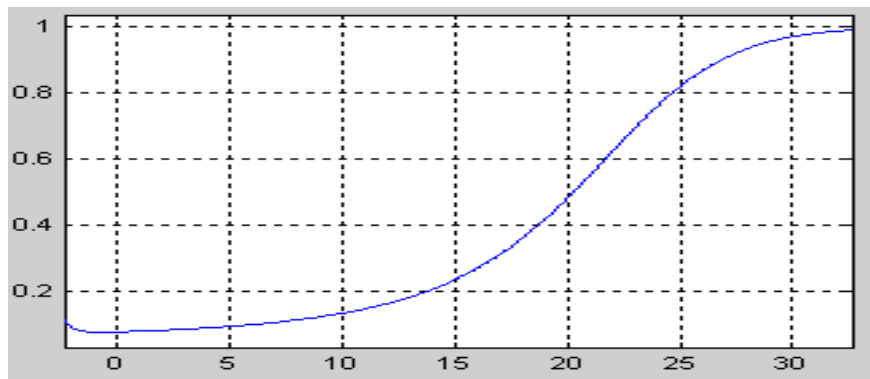
hence at $(1, 0)$ we have $\lambda = \frac{1}{2} \left[-c \pm \sqrt{c^2 - 4(1 - b)} \right]$

at $(f_0, 0)$ we have $\lambda = \frac{1}{2} \left[-c \pm \sqrt{c^2 + 4(b - f_0)} \right]$

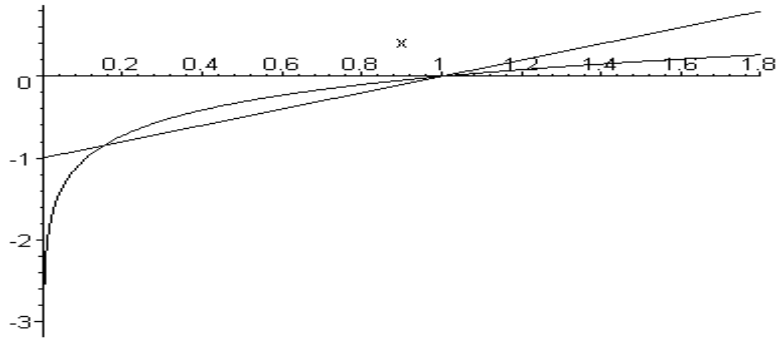
making $(f_0, 0)$ a saddle point for all values of c , while $(1, 0)$ is a stable focus when $c^2 < 4(1 - b)$ and a stable node when $c^2 > 4(1 - b)$. The stable focus leads to a heteroclinic orbit with both negative and positive values for g , which is not admissible when g denotes a population value. Then the physically reasonable solutions are associated with $c^2 > 4(1 - b)$, which defines a minimal speed for travelling wave solutions. The phase plane portrait in this case is as shown below



The heteroclinic orbit joining the saddle at $(f_0, 0)$ to the stable node at $(1, 0)$ is the travelling wave



Note that f_0 , as the solution of $b \log f = f - 1$, is an increasing function of the parameter $b = \frac{a}{rS_0}$,

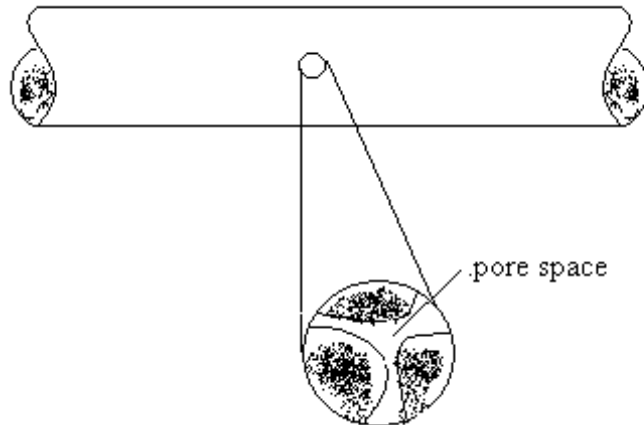


f-1, b log(f) versus f

Then as the death rate of infected foxes, a , increases or as the infection rate for the uninfected foxes, r , goes down the survival level f_0 goes up. This is exactly as you would suppose. On the other hand, f_0 , goes down as S_0 , the initial number of uninfected foxes goes up. This is less to be expected.

Transport of a Chemical Solute

Consider a tube of uniform cross section A containing a porous medium (soil) through which a fluid containing chemical solute is flowing. As the solute moves through the porous medium, some of the solute comes out of solution and sorbs to the soil. Conversely, chemical which is sorbed to the soil can dissolve in the solute and move away from the sorption site. We will derive a model for this process and examine it for existence of travelling wave solutions.



Let ϕ denote the porosity, the fraction of total volume that can be occupied by fluid. Then the state variables for describing the transport of chemical solute through the soil are as follows

$\vec{q}(x,t)$ = flux of solute at (x,t)

$C(x,t)$ = solute concentration at (x,t)

$N(x,t)$ = mass concentration of solute sorbed to the soil

Let (a, b) and (t_1, t_2) denote an arbitrary interval of the tube and an arbitrary interval of time during the process, respectively. Then the conservation of solute statement for this process is,

$$\int_a^b \phi C(x, t_2) A dx = \int_a^b \phi C(x, t_1) A dx + \int_{t_1}^{t_2} q(a, t) A dt - \int_{t_1}^{t_2} q(b, t) A dt - \int_{t_1}^{t_2} \int_a^b \partial_t N(x, t) dx dt.$$

In the usual way, this becomes

$$\int_{t_1}^{t_2} \int_a^b (\phi \partial_t C + \partial_x q + \partial_t N) dt dx = 0$$

and, since (a, b) and (t_1, t_2) are arbitrary,

$$\phi \partial_t C(x, t) + \partial_x q(x, t) + \partial_t N(x, t) = 0 \text{ at each } (x, t).$$

We suppose that solute flux is composed of two parts,

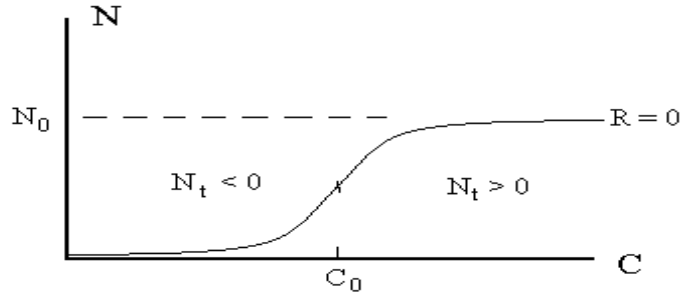
$$q(x, t) = \underbrace{-D \partial_x C(x, t)}_{\text{diffusion}} + \underbrace{\frac{1}{2} \phi B C(x, t)^2}_{\text{bulk transport}}$$

where the first term is the usual diffusion mechanism while the second part reflects transport by movement of the chemically contaminated fluid through the soil.

Finally, there are various ways in which, $N(x, t)$, the mass concentration of chemical that has been sorbed to the soil can be related to, $C(x, t)$, the concentration of chemical in solution. In order of increasing complexity, three possible choices are

- i) $N(x, t) = \alpha C(x, t)$
- ii) $\partial_t N(x, t) = \beta(N_0 - N(x, t))$
- iii) $\partial_t N(x, t) = R(N, C) = K \left(\frac{N_0 C^2}{C_0^2 + C^2} - N(x, t) \right)$

The first choice is mathematically simple but physically not very realistic. The second choice is more realistic in that it recognizes that the sorption rate must depend on the current value of N and it assumes there is some maximum value for the amount of solute that can be sorbed to the soil. Finally, the last choice treats the sorption rate as depending on the current value of both C and N . The following figure shows the curve where $R = 0$. To the left of this curve we have $\partial_t N(x, t) < 0$ and to the right, $\partial_t N(x, t) > 0$.



Then we have the following model for this process

$$\phi \partial_t C(x,t) - D \partial_{xx} C(x,t) + \phi B C(x,t) \partial_x C(x,t) + \partial_t N(x,t) = 0$$

$$\partial_t N(x,t) - K \left(\frac{N_0 C^2}{C_0^2 + C^2} - N(x,t) \right) = 0.$$

This is equivalent to

$$\partial_t \left(\frac{N}{N_0} \right) + K \left(\frac{N}{N_0} - \frac{(C/C_0)^2}{1 + (C/C_0)^2} \right) = 0,$$

$$\frac{1}{BC_0} \partial_t (C/C_0) - \frac{D}{\phi BC_0} \partial_{xx} (C/C_0) + (C/C_0) \partial_x (C/C_0) + \frac{N_0}{\phi BC_0^2} \partial_t (N/N_0) = 0$$

Reducing this to dimensionless form by letting

$$\sigma = \frac{C}{C_0}, \quad \eta = \frac{N}{N_0}, \quad z = x \frac{\phi BC_0}{D}, \quad \tau = \frac{\phi B^2 C_0^2}{D} t,$$

we obtain

$$\partial_\tau \sigma + \sigma \partial_z \sigma + \beta \partial_\tau \eta = \partial_{zz} \sigma$$

$$\partial_\tau \eta + k \left(\eta - \frac{\sigma^2}{1 + \sigma^2} \right) = 0,$$

where

$$\beta = \frac{N_0}{C_0 \phi}, \quad k = \frac{KD}{\phi BC_0^2}.$$

Now we suppose $\sigma(z, \tau) = f(z - v\tau)$, and $\eta(z, \tau) = g(z - v\tau)$

where

$$f, g \rightarrow 0 \text{ as } z \rightarrow \infty$$

$$f', g' \rightarrow 0 \text{ as } z \rightarrow -\infty$$

This amounts to assuming the concentrations of solute and sorbed chemical ahead of the wave are zero and are equal to unknown constants (to be determined) behind the wave. Then our system of equations reduces to

$$-vf' + ff' - \beta v g' = f''$$

$$-vg' + k\left(g - \frac{f^2}{1+f^2}\right) = 0.$$

It follows from integrating the first equation that

$$f'(\xi) = \frac{1}{2}f^2(\xi) - vf(\xi) - \beta v g(\xi)$$

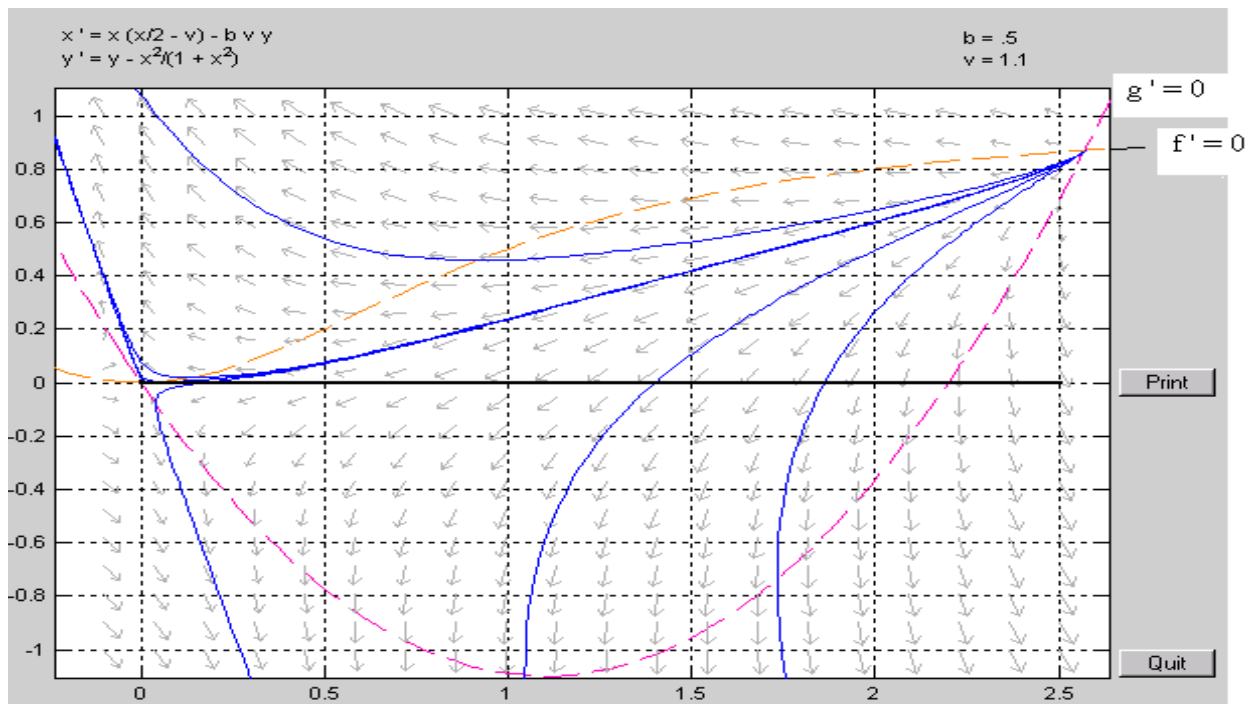
$$g'(\xi) = \frac{k}{v}\left(g - \frac{f^2}{1+f^2}\right).$$

Evidently this system has the following null isoclines

$$f' = 0 \quad \text{on} \quad g = \frac{f(f-2v)}{2\beta v} \quad (\text{a parabola})$$

$$g' = 0 \quad \text{on} \quad g = \frac{f^2}{1+f^2} \quad (R = 0)$$

The null isoclines are shown in the figure:



The isoclines meet at the origin, which is one of the critical points for the system. This critical point can be shown to be a saddle point. The other critical point is located where the null clines meet in the first quadrant; i.e.,

$$2\beta v f = (f - 2v)(1 + f^2)$$

or

$$f^3 - 2vf^2 + (1 - 2\beta v)f - 2v = 0.$$

For each $v > 0$, this cubic has a positive real root f_0 and then

$$(f_0, g_0) \text{ with } g_0 = \frac{f_0^2}{1 + f_0^2} \text{ is a critical point for the system.}$$

It would appear that there is a heteroclinic orbit joining (f_0, g_0) to $(0, 0)$ and such an orbit would be associated with a TW joining the state (f_0, g_0) at $\xi = -\infty$ to $(0, 0)$ at $\xi = \infty$. Since we cannot solve for (f_0, g_0) without explicit values for β, v we cannot determine the type of critical point (f_0, g_0) is. (For the values used to generate the figure, it is an unstable node). However, it is evident from the figure that the direction field of flow points **out** of the region W that is bounded on the sides by the two null clines and on the bottom by $g = 0$. This means that if we proceed backward in time along the heteroclinic orbit that arrives at $(0, 0)$ at $\xi = +\infty$, the direction of flow is **into** the region W and that the heteroclinic orbit can therefore not leave W through any of the three boundaries. In fact, since the side boundaries are null isoclines, the orbit cannot cross either of these without violating uniqueness. Then the orbit must tend to the critical point (f_0, g_0) as $\xi \rightarrow -\infty$, and the corresponding travelling waves then appear as follows

