4 PARTIAL DIFFERENTIAL EQUATIONS

In the work we have done so far we have only been concerned with the dependence of something on one variable, e.g. the deformation as a function of location, the number of animals or the concentration of some chemical species as a function of time. However, in many situations this is not enough information. In the case of the spread of a disease across the countryside, such as rabies in foxes across the UK, we would be interested not only in the number of infected animals, but also in their general location. In other words, as well as knowing the dependence on time, we would like to know how widely the disease has spread. Thus we must include spatial considerations in our models. To do this we need to consider functions of several variables, such as $f(x, y, t)$, where f depends on space (x, y) and time t. It is probably a good idea to revise the basic ideas of functions of several variables from previous courses at this point.

4.1 Basic considerations

An equation involving one or more derivatives of an unknown function of two or more variables is called a *Partial Differential Equation.* The most general pde of 1st order in two independent variables is

$$F(x, y, u, u_x, u_y) = 0.$$

Some pde occur repeatedly in different problems.

(i) $u_t + cu_x = 0$ – transport, traffic flow, waves in shallow water.

(ii) $u_{tt} = c^2 u_{xx}$ – 1-D wave eqn. – seismic, water, sound waves.

(iii) $u_t = \alpha u_{xx}$ – 1-D heat/diffusion eqn. – heat transfer, population or disease spread, modelling turbulence, spread of solute in solvent, groundwater flow...

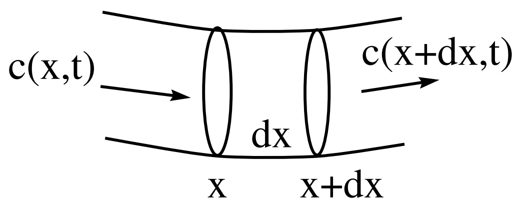
(iv) $u_{xx} + u_{yy} = f(x, y)$ – 2-D Poisson eqn. If $f(x, y) \equiv 0$ it is called Laplace's eqn. – Steady state temperature distribution, deflection of membranes, electrostatics, gravitation, fluid dynamics, groundwater flow.

(v) $u_t - iu_{xx} = 0$ ($i = \sqrt{-1}$) – quantum mechanics

(vi) $u_t + uu_x + u_{xxx} = 0$ – dispersive waves

Example

As an example of deriving a partial differential equation, we consider a simple model of the flow of blood cells in a vein or artery. We consider a small section of a circular tube of fixed cross-sectional area, A (cm^2), containing blood flowing with velocity $c(x, t)$, (cm/s) and with cell density $\rho(x, t)$ (cells/cc), where x (cm) is the distance along the vein. The change in the total number of blood cells within a section of the vein of length Δx can be calculated as the number of cells that flow in minus the number that flow out.



The number that flow in at x in time Δt is $\rho(x,t)c(x,t)A\Delta t$ while the number that flow out at $x+\Delta x$ is $\rho(x+\Delta x,t)c(x+\Delta x,t)A\Delta t$. Check that these have the right units, i.e. are a number. This must equal the difference in the total number of cells within at the two times, i.e we have

$$N(t+\Delta t) - N(t) = Inflow - Outflow$$
$$\rho(x,t+\Delta t)A\Delta x - \rho(x,t)A\Delta x = \rho(x,t)c(x,t)A\Delta t - \rho(x+\Delta x,t)c(x+\Delta x,t)A\Delta t$$

and dividing by $A\Delta t\Delta x$ gives,

$$\frac{\rho(x,t+\Delta t)-\rho(x,t)}{\Delta t} = \frac{\rho(x,t)c(x,t)-\rho(x+\Delta x,t)c(x+\Delta x,t)}{\Delta x}$$
$$\text{and letting } \Delta x\Delta t \to 0, \quad \frac{\partial \rho}{\partial t} = -\frac{\partial (c\rho)}{\partial x}$$
$$\text{or,} \quad \rho_t + c\rho_x + \rho c_x = 0,$$

and if c is constant this reduces to $\rho_t + c\rho_x = 0$. This is known as the advection or transport equation, and models particles that are carried along with the motion of a fluid. In fact it is a simple model only of these things, because much has been neglected, but it does show how an equation can be derived, and how a function can depend on both location and time.

In general, it is very difficult to solve pdes. It was hard enough to solve ODEs. There are some ways, but usually for simple geometry, e.g. rectangles, circles. Here we will cover several methods including separation of variables with Fourier series and numerical methods.

A general 2nd order pde has the form

$$A\phi_{xx} + 2B\phi_{xy} + C\phi_{yy} + D\phi_x + E\phi_y + F\phi + G = 0,$$

where A, B, C, D, E, F, G are all functions. If $G \equiv 0$, i.e. every term contains the dept. variable or one of its derivatives, the pde is homogeneous, otherwise it is inhomogeneous. If $A - G$ are functions of x and y only, then the equation is *linear*. If any of $D - G$ are functions of ϕ, ϕ_x, ϕ_y then the equation is *quasilinear*, i.e. linear at least in the highest order terms.

A linear equation has the property that if $\mathcal{L}$ is a differential operator, then

$$\mathcal{L}\{u+v\} = \mathcal{L}\{u\} + \mathcal{L}\{v\},$$

and also has the property that if we add two solutions to a given equation, then the sum is also a solution.

Example

Show that the equation $u_{tt} - u_{xx} + \epsilon u^2 = 0$ is linear if $\epsilon = 0$, and not linear otherwise.

Consider $u + v$,

$$\frac{\partial^2 (u+v)}{\partial t^2} - \frac{\partial^2 (u+v)}{\partial x^2} + \epsilon(u+v)^2 = \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial x^2} + \epsilon(u^2 + 2uv + v^2)$$

so that

$$\mathcal{L}\{u+v\} = \mathcal{L}\{u\} + \mathcal{L}\{v\} + \epsilon 2uv,$$

and so is linear provided $\epsilon = 0$.

The method that is used to solve a 2nd order pde is often dependent upon what type of pde it is. Second order, linear PDEs such as that above are classified as;

If $B^2 - AC = 0$, then the equation is said to be **parabolic**.

If $B^2 - AC > 0$, it is **hyperbolic**.

If $B^2 - AC < 0$, it is **elliptic**.

Example

The method of characteristics for a hyperbolic pde gives the so-called *d'Alembert's* solution to the wave equation, $\phi_{tt} - c^2\phi_{xx} = 0$, $\phi = f(x - ct) + g(x + ct)$. Show this is a solution.

$\phi_t = f'(x-ct).(-c)$ by the chain rule, and so $\phi_{tt} = c^2 f''(x-ct)$. Similarly, $\phi_x = f'(x-ct).1$, and also $\phi_{xx} = f''(x-ct)$, so $\phi_{tt} + \phi_{xx} = c^2 f''(x-ct) - c^2(f''(x-ct)) = 0$. The calculation for g is almost the same. Note this solution represents two signals, one moving to the left with velocity c and the other to the right with the same velocity. The initial condition is that at $t = 0$, $\phi(x, 0) = f(x) + g(x)$.

Notice that in this classification the wave equation is hyperbolic ($A = 1, B = 0, C = -1$, so $B^2 - AC = 1 > 0$), the heat/diffusion equation is parabolic ($A = \alpha, B = 0, C = 0$, so $B^2 - AC = 0$) and Laplace's equation is elliptic ($A = 1, B = 0, C = 1$, so $B^2 - AC = -1 < 0$).

These equations turn out to be *canonical* in the sense that any linear, 2nd order pde of each type can be transformed into the corresponding form. These equations all have constant coefficients but that is not necessary, and there are many situations in which the nature of the equation can change in different regions of space, i.e they may be elliptic in one region and parabolic in others.

Example

An example of these changes occurs in the *Tricomi equation*,

$$u_{\xi\xi} + \xi u_{\eta\eta} = 0.$$

The equation describes the transonic (near the speed of sound) flow of air over an aircraft. (This equation is derived using perturbation methods.) Note that if $\xi > 0$ then this equation is elliptic and if $\xi < 0$ it is hyperbolic reflecting different behaviour when the air is travelling faster or slower than the speed of sound. (It is parabolic if $\xi = 0$.)

If a pde has derivatives with respect to only one independent variable, then it can be solved as if it were an ordinary DE, but one must always keep in mind that the *constants* in the solution to an ODE must now be functions of the other independent variables.

Example

1. Solve $u_{xy} = 0$. Integrating with respect to x, $u_y = f(y)$, and then with respect to y, gives $u = \int f(y)dy + g(x) = h(y) + g(x)$. Note that $\int f(y)dy$ is just some function of y. Check by differentiating. Notice that integrating zero with respect to x gives a function of y, while integrating zero with respect to y gives a function of x.

2. The pde $u_{xx} - u = 0$, where $u = u(x, y)$ has the solution $u = f(y)e^x + g(y)e^{-x}$. This can be obtained as you would for a 2nd order ODE, i.e. let $u = e^{\lambda x}$ etc. but remember that the "constants" must be functions of y. Check by differentiating.

This means that the solution to a pde will in general have arbitrary functions in it, rather than constants (as for ODE). The values of these functions are determined by the boundary and initial conditions. Unlike ODEs, a linear pde may have many solutions. A unique solution is determined by the initial and boundary conditions. Implementing these initial and boundary conditions can sometimes be a bit confusing at first. In the next section we will see the geometric meaning of this, but here is an example just to see how it works.

Example

The general solution to $4u_x - 3u_y = 0$ is $u = f(-3x - 4y)$. If we know that $u = y^2$ when $x = 0$, what is the solution?

The boundary condition gives that $f(-4y) = y^2$, or $f(w) = (-\frac{w}{4})^2$, (let $w = -4y$), and so the solution is $u = f(-3x - 4y) = (-\frac{(-3x-4y)}{4})^2 = (\frac{3}{4}x + y)^2$.

Checking: we see $u_x = 2(\frac{3}{4}x + y)\frac{3}{4}$, and $u_y = 2(\frac{3}{4}x + y).1$ so that $4u_x - 3u_y = 6(\frac{3}{4}x + y) - 6(\frac{3}{4}x + y) = 0$, and also $u(0, y) = y^2$. We will do more of these as we go along.

Exercises

1. Consider the derivation of the advection equation given at the start of this section. Check that you understand each term and work out the dimensions of each term and check that they match. How would the derivation change if the area of cross-section, A, were also a function of x and t, i.e. $A(x, t)$?

2. Which of the following operators are linear?

(a) $\mathcal{L}u = u_x + xu_y$ (b) $\mathcal{L}u = u_x + uu_y$ (c) $\mathcal{L}u = u_x + u_y^2$

3. Solve the following pde, assuming the solution is a function of (x, t)

(a) $u_{xx} = 0$ (b) $tu_t = 2$ (c) $u_{tt} + u = 0$

4. Show by substituting that the given functions are solutions to the given equations.

(a) $au_x + bu_y = 0, u(x, y) = f(bx - ay)$ (b) $u_x + yu_y = 0, u(x, y) = f(ye^{-x})$

5. The solution to $u_t + 3u_x = 0$ is $u = f(x - 3t)$. Show this is true, and then find the solution with the initial condition $u(x, 0) = \sin 2x$.

6. Classify the partial differential equations as hyperbolic, elliptic or parabolic,

(a) $2u_{xx} - 2u_{xy} + 5u_{yy} + 3u_x - u + 1 = 0$ (b) $4u_{xx} + u_{xy} - u_{yy} - u + 4xy = 0$.

Solutions

2 (a) Linear (b),(c) Nonlinear

3 (a) $u(x, t) = xf(t) + g(t)$ (b) $u(x, t) = 2 \ln t + f(x)$ (c) $u = f(x) \sin t + g(x) \cos t$

5 $f(w) = \sin 2w \Rightarrow u(x, t) = \sin 2(x - 3t)$.

6 (a) elliptic (b) hyperbolic

4.2 Linear first-order equations

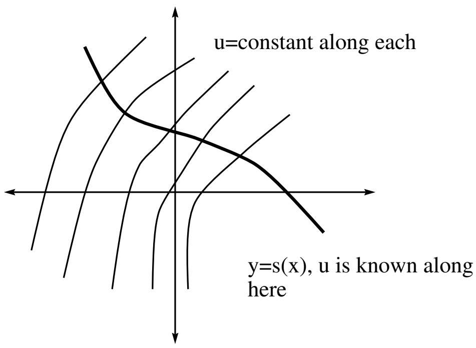
We will begin by considering 1st-order partial differential equations. The book by Strauss, *Partial Differential Equations - An introduction* might be useful if you want another source here. In principle, it is possible to get a solution to any quasilinear first-order pde. Once we know something about equations of this type, we will use them to derive and analyse traffic flow, an important problem in design of the urban environment.

Suppose we wish to solve the homogeneous, linear, 1st-order pde given by

$$f(x,y)u_x + g(x,y)u_y = 0, \text{ with } u = r(x,y) \text{ on } y = s(x).$$

The condition $u = r(x,y)$ on $y = s(x)$ is a general boundary or initial condition, such as $u(x,y) = y^2$ on $x = 0$. We will look at the general case first.

Suppose there is a special set of curves in the xy-plane on which the solution $u = u(x,y)$ is equal to a constant (this is almost certain to be true). If we can find this special set of curves, then we can use it as a new coordinate (and the problem would then only have one independent variable). If u is known on $y = s(x)$, and the lines on which u is constant are known, then it should be possible to find the values of u everywhere – see the figure below.



Since u is known along $y = s(x)$, then from the intersection points, it must be known along all of the other curves, so it is known everywhere. This means that if we can find the set of curves on which u is constant, then we have solved the pde. These curves are called *Characteristics* and are very important in obtaining solutions for these and higher order pde's.

Consider the chain rule applied to a function $u(x(t), y(t))$, where t is a parameter, then if u is constant along the curve parameterised by $(x(t), y(t))$, then

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} = x'(t)u_x + y'(t)u_y = 0.$$

This is a generalisation of the chain rule in one dimension. If we compare this with the first-order pde of the form,

$$f(x,y)u_x + g(x,y)u_y = 0,$$

we see that they are the same if

$$x'(t) = f(x,y) \text{ and } y'(t) = g(x,y) \Rightarrow \frac{y'(t)}{x'(t)} = \frac{g(x,y)}{f(x,y)},$$

an ODE for $y(x)$. In other words the solution to this DE is the set of curves on which $u(x,y)$ is constant (since $\frac{du}{dt} = 0$), i.e the characteristics.

Example

Solve the pde $u_x - 2u_y = 0$ subject to the condition that $u = x^3$ on $y = 0$.

Using the above results we see that $f = 1$ and $g = -2$ so the characteristics are given by

$$\begin{aligned}\frac{dy}{dx} &= \frac{g}{f} = -2\\ \Rightarrow dy &= -2dx\\ \Rightarrow y &= -2x + C\\ \text{i.e. } \beta &= y + 2x = \text{a constant}\end{aligned}$$

so the lines which satisfy $\beta = y + 2x$ for different β are the characteristics, i.e. u =const. along these lines, so the general solution must be of the form $u = f(\beta) = f(y + 2x)$. You can check this by substituting; $u_x = 2f'(\beta)$ and $u_y = f'(\beta)$ using the Chain Rule, so $u_x - 2u_y = 0$ checks. Now we also know that $u = x^3$ when $y = 0$, so $f(2x) = x^3 \Rightarrow f(\beta) = (\frac{\beta}{2})^3$, which means that the solution is $u(x, y) = \frac{1}{8}(y + 2x)^3$.You should check by substituting that this is the correct answer.

Here is a more complicated example, but the method is exactly the same.

Example

Solve the pde $u_x + xy^2u_y = 0$ with the condition $u = x^2$ on $y = 1$.

Using the above results, we see that $f = 1$ and $g = xy^2$, so the characteristics are given by

$$\begin{aligned}\frac{dy}{dx} &= \frac{g}{f} = xy^2\\ \Rightarrow \frac{dy}{y^2} &= xdx\\ \Rightarrow -y^{-1} &= \frac{1}{2}x^2 + C\\ \text{i.e. } \beta &= \frac{1}{2}x^2 + y^{-1} = \text{a constant}\end{aligned}$$

These are the characteristic curves, and that means that $u = u(\beta)$. This is where it gets a bit tricky. In effect, we have that u depends on $\beta = \frac{1}{2}x^2 + y^{-1}$, i.e. the general solution is

$$u = f(\frac{1}{2}x^2 + y^{-1}).$$

Now we also know that when $y = 1$, $u = x^2$, so we should be able to determine the function, i.e.

$$u = f(\frac{1}{2}x^2 + y^{-1})|_{y=1} = f(\frac{1}{2}x^2 + 1) = x^2,$$

which means that the function f operates by subtracting 1 and then multiplying by 2 (or let $\alpha = x^2/2 + 1 \Rightarrow x^2 = 2(\alpha - 1) \Rightarrow f(\alpha) = 2(\alpha - 1)$), i.e.

$$u = 2\left[\left(\frac{1}{2}x^2 + y^{-1}\right) - 1\right],$$

i.e, the solution is

$$u(x, y) = x^2 + \frac{2}{y} - 2.$$

We should check this;

$$u_x = 2x, \quad u_y = \frac{-2}{y^2} \Rightarrow u_x + xy^2u_y = 2x - \frac{2xy^2}{y^2} = 0 \text{ correct; and on } y = 1, u = x^2.$$

Exercises

1. Show that the general solution of the pde $au_x + bu_y = 0$, where a, b are constants, is $u = f(bx - ay)$, either by substituting or directly finding it as above.

2. Find the general solution to the following linear, homogeneous pde's.

(a) $w_t + xw_x = 0$ (b) $2xu_x + u_y/y = 0$

3. If $u = f(x, y)$ is the general solution to a pde, find the particular solution for the boundary information given.

(a) $u = f(3x - y^2)$, $u = y^2$ on $x = 1$ (b) $u = f(x^2 - y)$, with $u = 2\cos y$ on $x = 0$.

4. Solve the following equations with the given conditions.

(a) $u_x + 2xy^2u_y = 0$ with $u = y$ on $x = 0$ (b) $yu_x + xu_y = 0$ with $u(0, y) = e^{-y^2}$.

Solutions

2 (a) $w = f(e^{-t}x)$ (b) $u = f(y^2 - \ln x)$

3 (a) $u = 3 + y^2 - 3x$ (b) $u = 2\cos(y - x^2)$

4 (a) $u = y/(yx^2 + 1)$ (b) $u = e^{x^2 - y^2}$

4.3 Nonhomogeneous equations.

The model of traffic flow that we will derive unfortunately does not turn out to be linear. How can we extend our method to the more general nonhomogeneous and quasilinear cases? The general quasilinear equation is

$$f(x, y, u)u_x + g(x, y, u)u_y = h(x, y, u), \text{ with } u = r(x, y) \text{ on } y = s(x).$$

The answer to this question is to notice that if we have the solution $u = \phi(x, y)$, it will be a surface in x, y, u space. Differentiating with respect to t, we obtain

$$\frac{du}{dt} = \frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{dy}{dt}.$$

This is satisfied in the partial de if

$$\frac{du}{dt} = h(x, y, u), \quad \frac{dx}{dt} = f(x, y, u) \text{ and } \frac{dy}{dt} = g(x, y, u),$$

so, for a change in t, i.e. $dt = \dfrac{du}{h} = \dfrac{dx}{f} = \dfrac{dy}{g}$. $(*)$

This gives a set of curves which lie within the solution surface. The projections of these *characteristic* curves in (x, y, u) space onto the xy-plane are called the *characteristic traces*. We can satisfy the conditions in (*) in most cases by solving one pair and then eliminating the other variable. The best way to see this is to do an example.

Example

Solve the nonhomogeneous pde $xu_x + yu_y = 1 + y^2$ given that $u(x, 1) = x + 1$.

Using the above, we have that

$$\frac{du}{1+y^2} = \frac{dx}{x} = \frac{dy}{y},$$

so using the last two, we get that $y = \beta x$, for some constant β. In other words, the characteristic traces are some function of $\beta = y/x$. We can now note that

$$\frac{du}{1+y^2} = \frac{dy}{y} \Rightarrow du = \frac{(1+y^2)}{y}\, dy \Rightarrow u = \ln y + \frac{1}{2}y^2 + C(\frac{y}{x}).$$

It is very important to notice that C depends on y/x. The reason for this is that we are integrating along a characteristic trace, on which $y = \beta x$, so if there is a function of y/x it will not appear.

We must now use this and the extra condition $u(x, 1) = x + 1$ to determine the value of u. If $y = 1$, then

$$u = x + 1 = 0 + \frac{1}{2} + C(\frac{1}{x}) = x + 1$$

so that $C(\frac{1}{x}) = x + 1 - \frac{1}{2} = x + \frac{1}{2}$, and hence (let $\alpha = 1/x \Rightarrow x + \frac{1}{2} = \frac{1}{\alpha} + \frac{1}{2}$),

$$C(\frac{y}{x}) = \frac{x}{y} + \frac{1}{2}$$

and therefore

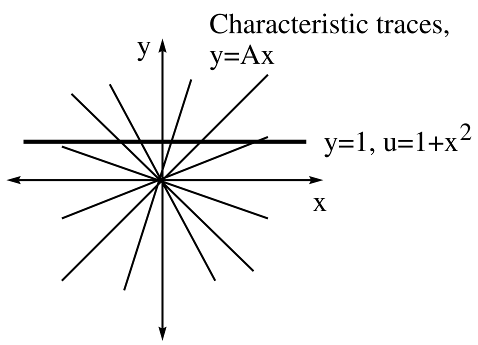
$$u(x, y) = \ln y + \frac{1}{2}y^2 + \frac{x}{y} + \frac{1}{2}.$$

Once again, we should check that we have the correct solution.

$$\begin{aligned} u_x = \frac{1}{y}, \quad u_y &= \frac{1}{y} + y - \frac{x}{y^2}, \\ \Rightarrow x u_x + y u_y &= \frac{x}{y} + 1 + y^2 - \frac{x}{y} = 1 + y^2, \\ \text{and} \quad u(x, 1) &= 0 + \frac{1}{2} + x + \frac{1}{2} = x + 1 \end{aligned}$$

and satisfies the equation and condition.

To solve this we have done something very similar to the earlier example. We have found a set of curves on which u is known, and then used the boundary condition to obtain values on those particular curves, as shown below.



The order in which you choose to integrate these equations is not important, in other words since we are working with surfaces, it is almost like (x, y, u) are three coordinates, rather than $u(x, y)$, i.e. u being a function of x and y.

Example

Solve the nonhomogeneous pde $u_x + 2xu_y = y$ given that $u = 1$ on $y = x^2$. We have

$$\frac{dx}{1} = \frac{dy}{2x} = \frac{du}{y} \Rightarrow 2xdx = dy \text{ or } y - x^2 = \beta$$

then

$$\frac{du}{x^2 + \beta} = dx \Rightarrow u = \frac{1}{3}x^3 + \beta x + f(\beta) = \frac{1}{3}x^3 + (y - x^2)x + f(y - x^2)$$

and so when $y = x^2$, $u = 1 \Rightarrow 1 = \frac{1}{3}x^3 + f(0)$. This can not be satisfied in general. If we look at the boundary condition, we can see why this is so – the boundary condition is applied along one of the characteristic traces !! The only way this can work is if the boundary condition matches exactly the behaviour along the characteristic trace given by the equation, otherwise there is no solution. If it does match, then there are an infinite number of solutions.

Example

Consider the same problem as above, but with the boundary condition $u = \frac{1}{3}x^3 + \pi$ on $y = x^2$. The general solution is as before, i.e. $u = \frac{1}{3}x^3 + (y - x^2)x + f(y - x^2)$, and the boundary condition gives $f(0) = \pi$. In this case, the boundary condition can be satisfied, but anything for which $f(\beta = 0) = \pi$ is an acceptable solution. In other words there are an infinite number of solutions !!

This means one should be very careful if the boundary data is specified along one of the characteristic traces, since no information is available to go to the other characteristics. There are sometimes ways around this, but we will not cover them in this course.

Example

Solve the nonhomogeneous pde $yu_x + u_y = x$ subject to $u(x, 0) = x^2$.

$$\begin{aligned}
&\frac{dx}{y} = \frac{dy}{1} = \frac{du}{x} \Rightarrow dx = y\ dy \Rightarrow x = \frac{1}{2}y^2 + \beta \\
\Rightarrow &x - \frac{1}{2}y^2 = \beta \Rightarrow dy = \frac{du}{\beta + \frac{1}{2}y^2} \Rightarrow (\beta + \frac{1}{2}y^2)\ dy = du \\
\Rightarrow &u = \beta y + \frac{1}{6}y^3 + f(\beta)
\end{aligned}$$

so that the general solution is (noting $\beta = x - y^2/2$),

$$u(x, y) = xy - \frac{1}{2}y^3 + \frac{1}{6}y^3 + f(x - \frac{1}{2}y^2).$$

Applying the initial condition, we obtain that

$$u(x, 0) = x^2 = 0 - 0 + 0 + f(x) \Rightarrow f(x) = x^2$$

so that the particular solution is

$$u(x,y) = xy - \frac{1}{3}y^3 + (x - \frac{1}{2}y^2)^2.$$

Checking: $u(x,0) = x^2$ checks, and $yu_x = y^2 + 2y(x - \frac{1}{2}y^2),\ u_y = x - y^2 - 2(x - \frac{1}{2}y^2)y$ and therefore $yu_x + u_y = x$ checks.

The quasi-linear case is much more convoluted than the nonhomogeneous case, and often the best we can do is get an implicit form for the solution. We will not consider them in this course, but the procedure is the same.

Exercises

1. Solve the following problems.

(a) $u_x + (x-1)u_y = u$ subject to $u = e^{2x}$ on $x + y = 0$.

(b) $u_x + 2xu_y = y$ with $u(0,y) = 1 + y^2$.

(c) $x^2u_x + yu_y = x^2$ with $u = 3x$ on $y = 1$

Solutions

(a) $u(x,y) = \exp(x + \sqrt{x^2 - 2y - 2x})$

(b) $u(x,y) = yx - \frac{2}{3}x^3 + 1 + (y - x^2)^2$

(c) $u(x,y) = x + \frac{2}{(\ln y + \frac{1}{x})}$

4.4 A model for road traffic flow

This example comes from the book *An Introduction to Mathematical Modelling* by Fowkes and Mahoney. This section is very difficult and you will need to work through it very carefully to understand what is going on.

Coming up with a model for traffic flow can be useful in planning streets or deciding on speed limits which will clear a city block most quickly. It is an integral part of urban environment design. We need to come up with a model that is simple enough to give some answers without unnecessary hassles, but that has enough detail to reflect the true situation.

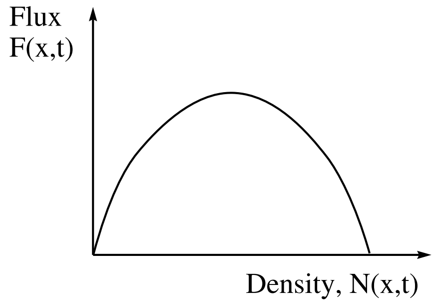
One way to start is to consider two particular points on the road and see how many cars go past, and then derive equations accordingly. Obviously we have to choose sufficient distance between the points for this to make sense. We want to avoid having distortions due to the motions of individual cars, and concentrate on the general traffic flow. This is a bit like the way in which fluid flow is modelled. We assume that the fluid is a continuum rather than a bunch of molecules, so we consider volume elements that are large enough to be a continuum, but small enough to apply the ideas of calculus. The end result in this case should be a continuous model rather than a discrete model (they are usually easier to work with).

First, we need to define a set of variables. Let $N(x,t)$ be the density of cars (no. cars per metre) at any location. Let $F(x,t)$ be the flux of cars past a particular point (no. cars per time). These variables satisfy the relation $F = NV$, where V (m/s) is the velocity of the flow of traffic. Our own experiences tell us that there is some relationship between the two functions F and N, i.e. if the traffic is really dense, it tends to move move slowly. We also know that if the density is extremely small, then the flow past a particular point will be small, and approach zero as the density approaches zero.

Consider that as the speed increases, the distance between cars increases (for safety), so the density would be a decreasing function of speed, i.e. in general less cars in a given distance if travelling at higher speeds. We also suggest that drivers will generally travel at some maximum speed if there is little traffic, i.e. $V \to V_{max}$ as $N \to 0$, probably just over the local speed limit. Finally, if the distance between cars is very small, then drivers will hardly dare move, i.e. $V \to 0$ as $N \to N_{max}$. A trial version of this relationship might be that $V(N) = V_{max}(1 - N/N_{max})$.

Given that $F = NV = NV(N)$, an equation that might be used is

$$F(N) = V_{max}N(1 - N/N_{max}).$$



This model is consistent with what we observe. If the density of traffic is small, the flux will be small (since there are few cars), despite the speed being high, while if the traffic density is high, the cars will be travelling slowly, so once again the flux will be small.

There will be some point at which the flux will be a maximum, and this will correspond to some particular density, N^* (say). This maximum will depend on the situation, e.g the number of lanes, the weather conditions etc. At any other flux, there will be two possible solutions, one with high velocity and lower density, and one with lower velocity and higher density. Obviously, the preferred option of the two is the one at the higher speed with lower density, but it is not clear which situation will arise. It seems likely that the solution obtained will depend on the initial conditions. For example, if the traffic has been heavy, then the slower branch is more likely, but if it has recently been running smoothly, then it would seem likely that the faster travel, lower density situation might arise. It is clear then that we need to examine the unsteady situation in order to determine which final state (if any) will be reached.

The model we have here is flawed in another way. At present the maximum flux occurs when the velocity is one half of the maximum velocity, which seems particularly unlikely. There is no reason why the optimum flow should occur at one half of an arbitrarily chosen speed limit. One way to correct this is to take measurements of what this optimum value is and then skew the flux vs. density relationship accordingly. This can be done by introducing a cubic term and using its coefficient to match the optimum with the measured value. This could get messy, so we will proceed with what we have - we should still be able to reproduce some of the qualitative behaviour. Lets see how we get on.

Unsteady Model

To consider how the situation changes with time, consider the flow of traffic between two points on the freeway, say $x = a$ and $x = b$. The change in the number of cars between a and b is

$$\frac{\partial}{\partial t}\int_a^b N(x,t)\ dx,$$

and this must match the flow into this region minus the flow out, i.e.

$$F(a,t) - F(b,t) = -\int_a^b F_x(x,t)\ dx.$$

Equating gives

$$N_t + F_x = 0 = N_t + F'(N)N_x,$$

noting that $F(N)$ and using the Chain rule. This is our unsteady model for traffic flow. We can try looking at the behaviour of this model for different functions $F(N)$.

The first thing to notice is that $F'(N)$ has the units of a velocity. In fact this equation is very similar to the one-dimensional advection equation,

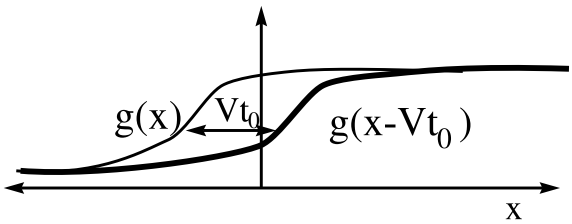
$$\rho_t + c\rho_x = 0,$$

which models the transport of some pollutant (or some other quantity) which has been released in a stream, where ρ is the concentration and c is the speed of flow of the stream. Notice that there is no diffusion term, i.e. $\alpha\rho_{xx}$. This is because for such a case the amount of diffusion would be very small compared to the *advection* unless c was very small.

Getting back to this case, the form of $F'(N)$ will dictate the behaviour of the traffic. For example, consider what will happen if $F'(N) = V$ where V is a constant. In that case, the equation is $N_t + VN_x = 0$, which by the method of characteristics has the solution

$$\frac{dt}{1} = \frac{dx}{V} \Rightarrow x - Vt = \beta \Rightarrow N = N(x - Vt).$$

To see how this behaves, consider an initial condition that $N(x) = g(x)$ where $g(x)$ is as shown below.

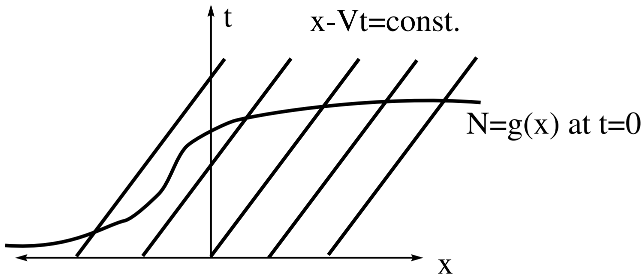


This represents a region where there is an increase in density at some point, e.g where some traffic has caught up with cars leaving some lights, or where some traffic ahead has just merged lanes. The second curve shows the solution a time t_0 later. In other words if V is a constant, then the original signal just propagates along the freeway retaining its shape all of the way. This is what you would expect if you think about it - if all of the traffic is moving at the same speed, then the original density function will just remain, but move along with that speed.

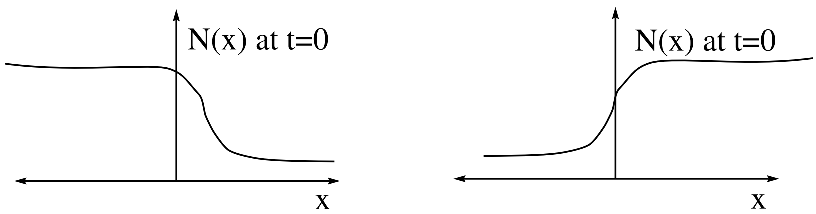
However, our discussion above lead us to a different form for the velocity as a function of the density. Consider for a moment the general case, where $F'(N)$ is any function. What can we determine about the flow ? This is a quasilinear equation, since N appears in the coefficient of N_x. Recall our initial derivation of the solution of such a homogeneous equation. It suggested that we have N constant on the characteristic curves, in this case given by

$$\frac{dx}{dt} = F'(N).$$

If we can integrate this equation, we can obtain the lines along which N is constant in the xt-plane, i.e. the space time curves on which the density does not change. Note that if $F'(N)$ is a constant, then the curves are simply the straight lines $x - Vt = \beta$, and would appear as below if sketched in the xt-plane.



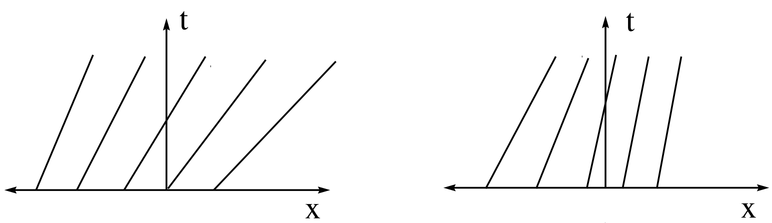
Once the starting density is specified, we can find the value at any later time at a particular location by tracing the characteristic back to the initial time. If the value of $F'(N)$ is known at the starting time at a particular point x, then it will equal $x'(t)$, and since N is constant along this characteristic, it will not change, and hence $F'(N)$ will not change as we move along it, so the slope will not change. This means that the characteristics will be straight lines that have slope $F'(N)$, which is determined by the starting value. Consider the following two examples.



(a) (b)

If we start with (a) the first situation (traffic density decreasing, e.g just after merging at Mill Point entry), then if we examine our density vs. flux relation, we see that as N decreases, the slope of $F(N)$,i.e. $F'(N)$, increases, so that $\frac{dt}{dx} = 1/F'(N)$ decreases. Thus the characteristic curves will be straight lines with slopes shown as in (a) below, provided we are to the left of the maximum flux case.

On the other hand, in the second case (b), just approaching Mill Point road before the merging (say), then the slope $F'(N)$ will be decreasing as the density increases (again if we are to the left of the maximum flux case). The characteristics in this case will look as in (b) below.

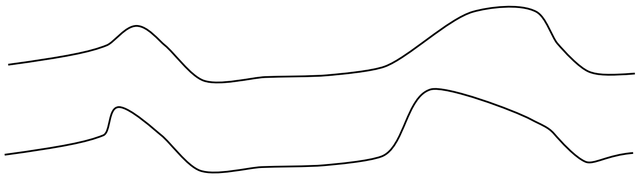


(a) (b)

If the density is greater than the maximum flux situation then these plots would be reversed. A plot of density at any time can then be calculated by drawing a horizontal line across these plots, and working out the density values of each. If the characteristics are spreading out, as in (a), this means that the density step is smoothing out as cars accelerate away. However, if the characteristics are closing together, as in (b), this means that the density step is steepening up. If we continue high enough up the t axis, these lines will eventually cross, which corresponds to multiple values of density, i.e. the step has steepened until it is vertical. This is a *shock wave*. What it means physically is difficult to interpret from this model, because this model will break down here. However, there are ways that analysis can be performed on either side of the shock. Unfortunately, we don't have time to do this.

Comments

1. The model we have derived is a very simple one, but nonetheless is able to reproduce most of the behaviour that we witness in traffic flow situations. It is a reasonable basis for a more sophisticated model.
2. The method we have used of examining the solution by following the characteristics is a commonly used numerical technique for such equations, and also for second-order hyperbolic pde.
3. Using the discussion above, we can come up with a general idea of the traffic behaviour given some starting pattern of traffic density, perhaps as shown in the upper line below.



A short time later, see the lower line above, places where the density decreases as you move in the direction of flow will spread out, while those where it increases in the direction of flow will steepen, so it looks a little like waves coming into the shore, but backwards (a situation modelled by another, higher order pde which has similar properties). This would suggest that a shock will always form, but other factors not modelled become significant, and a modification to the model must be made. However, if the density increase becomes large enough, then a shock will form. Lets consider a few cases.

If the traffic flow is reasonably clear, the steepening curve will probably clear the network before it becomes a factor, since the traffic speed is high. As the density increases, the slope of $F(N)$ decreases, then the disturbance will travel more slowly, and so will affect more and more cars. If the density reaches the value of maximum flux, then the value of $F'(N)$ will be zero, so the disturbance will remain at a fixed location - a disaster !! Every car approaching it will be affected. This situation is particularly nasty because it occurs at precisely the desirable flux rate ! Finally, if N becomes greater than the maximum flux, then $F'(N)$ becomes negative, and the disturbance propagates backwards into the traffic stream at quite a high speed, so that a minor disruption ahead can cause a major disruption to the oncoming cars.

More situations are considered in *Fowkes and Mahoney* if you are interested. They consider flow away from traffic lights and traffic jams in more detail, and also examine the shock waves in more detail. We must move on to the next section, with a note that the ideas of signal propagation examined in this example are relevant in a number of real situations. Shock waves form in many other circumstances, such as supersonic flow (sonic boom), flow of water over weirs (hydraulic jumps), etc....

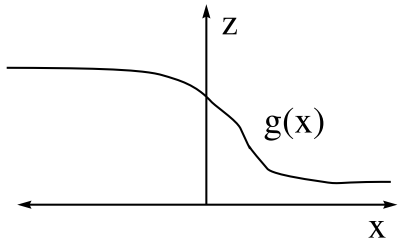
Exercises

1. This is a very difficult problem. You will need to work through it very carefully. Consider the equation

$$\frac{\partial z}{\partial t} + z\frac{\partial z}{\partial x} = 0.$$

This equation describes the evolution of $z(x,t)$, the free-surface elevation in a shallow stream. Suppose also that $z = g(x)$ at $t = 0$ is as shown below, i.e. this is the shape of the water surface at $t = 0$.

(i) Obtain an expression for the characteristic emanating from some point x_0, and show that the slope in the $x - t$ plane is given by $\frac{dt}{dx}|_{x_0} = 1/g(x_0)$, and hence sketch a set of characteristics for the function $g(x)$ given below.



(ii) Show that $z = f(x - zt)$ is the general solution to this equation (be very careful here - remember that z depends on x and t). Find z in terms of g given the above initial condition. Consider the form of z_x, and comment on the change in shape of the free surface which you would expect as time goes on, giving your reasoning. Is there some critical event, and at approximately what time would you expect it to occur?

Solution

(i) $g(x_0)$ is a constant for given x_0 so characteristics can be written as $x = g(x_0)t + C$, a set of straight lines with slope $g(x_0)$. Remember $z = \textit{constant}$ along these lines, so these lines will have constant slope, and since $1/g(x_0)$ increases, the slope will increase.

(ii) $z_t = f'(x - zt)(-z - z_t t)$ by the Chain Rule etc. Also, $z = g(x)$ at $t = 0$ means $f = g$. The wave will steepen and break when

$$z_x = \frac{f'(x - zt)}{1 + f'(x - zt)t}$$

becomes singular, i.e $t_c = -1/f'(x - zt)$.